COMBINATORIAL MARKOV CHAINS ON LINEAR EXTENSIONS

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Abstract. We consider generalizations of Schützenberger's promotion operator on the set $\mathcal{L}$ of linear extensions of a finite poset of size n. This gives rise to a strongly connected graph on $\mathcal{L}$. By assigning weights to the edges of the graph in two different ways, we study two Markov chains, both of which are irreducible. The stationary state of one gives rise to the uniform distribution, whereas the weights of the stationary state of the other has a nice product formula. This generalizes results by Hendricks on the Tsetlin library, which corresponds to the case when the poset is the antichain and hence $\mathcal{L} = S_n$ is the full symmetric group. We also provide explicit eigenvalues of the transition matrix in general when the poset is a rooted forest. This is shown by proving that the associated monoid is $\mathcal{R}$-trivial and then using Steinberg's extension of Brown's theory for Markov chains on left regular bands to $\mathcal{R}$-trivial monoids.

1. Introduction

Schützenberger [Sch63] introduced the notion of evacuation and promotion on the set of linear extensions of a finite poset P of size n. This generalizes promotion on standard Young tableaux defined in terms of jeu-de-taquin moves. Haiman [Hai92] as well as Malvenuto and Reutenauer [MR94] simplified Schützenberger's approach by expressing the promotion operator ∂ in terms of more fundamental operators τ_i $(1 \leq i < n)$, which either act as the identity or as a simple transposition. A beautiful survey on this subject was written by Stanley [Sta09].

In this paper, we consider a slight generalization of the promotion operator defined as $\partial_i = \tau_i \tau_{i+1} \cdots \tau_{n-1}$ for $1 \leq i \leq n$ with $\partial_1 = \partial$ being the original promotion operator. Since the operators ∂_i act on the set of all linear extensions of P, denoted $\mathcal{L}(P)$, this gives rise to a

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graph whose vertices are the linear extensions and edges are labeled by the action of ∂_i. We show that this graph is strongly connected (see Proposition 4.1). As a result we obtain two irreducible Markov chains on $\mathcal{L}(P)$ by assigning weights to the edges in two different ways. In one case, the stationary state is uniform, that is, every linear extension is equally likely to occur (see Theorem 4.3). In the other case, we obtain a nice product formula for the weights of the stationary distribution (see Theorem 4.5). In this paper, we also consider analogous Markov chains for the adjacent transposition operators τ_i, and give a combinatorial formula for their stationary distributions (see Theorems 4.4 and 4.7).

Our results can be viewed as a natural generalization of the results of Hendricks [Hen72, Hen73] on the Tsetlin library [Tse63], which is a model for the way an arrangement of books in a library shelf evolves over time. It is a Markov chain on permutations, where the entry in the ith position is moved to the front with probability p_i. Hendricks' results from our viewpoint correspond to the case when P is an antichain and hence $\mathcal{L}(P) = S_n$ is the full symmetric group. Many variants of the Tsetlin library have been studied and there is a wealth of literature on the subject. We refer the interested reader to the monographs by Letac [Let78] and by Dies [Die83], as well as the comprehensive bibliographies in [Fil96] and [BHR99].

One of the most interesting properties of the Tsetlin library Markov chain is that the eigenvalues of the transition matrix can be computed exactly. They were independently investigated by several groups, notably Donnelly [Don91], Kapoor and Reingold [KR91], and Phatarfod [Pha91] studied the approach to stationarity in great detail. There has been some interest in finding exact formulas for the eigenvalues for generalizations of the Tsetlin library. The first major achievement in this direction was to interpret these results in the context of hyperplane arrangements [Bid97, BHR99, BD98]. This was further generalized to a class of monoids called left regular bands [Bro00] and subsequently to all bands [Bro04] by Brown. This theory has been used effectively by Björner [Bjö08, Bjö09] to extend eigenvalue formulas on the Tsetlin library from a single shelf to hierarchies of libraries.

In this paper we give explicit combinatorial formulas for the eigenvalues and multiplicities for the transition matrix of the promotion Markov chain when the underlying poset is a rooted forest (see Theorem 5.2). This is achieved by proving that the associated monoid is $\mathcal{R}$-trivial and then using a generalization of Brown's theory [Bro00] of Markov chains for left regular bands to the $\mathcal{R}$-trivial case using results of Steinberg [Ste06, Ste08].

Computing the number of linear extensions is an important problem for real world applications [KK91]. For example, it relates to sorting algorithms in computer science, rankings in the social sciences, and efficiently counting standard Young tableaux in combinatorics. A recursive formula was given in [EHS89]. Brightwell and Winkler [BW91] showed that counting the number of linear extensions is $\#P$-complete. Bubley and Dyer [BD99] provided an algorithm to (almost) uniformly sample the set of linear extensions of a finite poset quickly. We propose new Markov chains for sampling linear extensions uniformly randomly. Further details are discussed in Section 7.

The paper is outlined as follows. In Section 2 we define the extended promotion operator and investigate some of its properties. The extended promotion and transposition operators are used in Section 3 to define various Markov chains, whose properties are studied in Section 4. In Section 5 we derive the partition function for the promotion Markov chains for rooted forests as well as all eigenvalues together with their multiplicities of the transition matrix. The statements about eigenvalues and multiplicities are proven in Section 6 using the theory of $\mathcal{R}$-trivial monoids. We end with possible directions for future research in Section 7. In Appendix A we provide details about implementations of linear extensions, Markov chains, and their properties in `Sage` [S$^+$12, SCc12] and `Maple`.

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2. Extended promotion on linear extensions

2.1. **Definition of extended promotion.** Let P be an arbitrary poset of size n, with partial order denoted by $\preceq$. We assume that the vertices of P are labeled by elements in $[n] := \{1, 2, \ldots, n\}$. Let $\mathcal{L} := \mathcal{L}(P)$ be the set of its **linear extensions**,

$$\mathcal{L}(P) = \{\pi \in S_n \mid i \prec j \text{ in } P \implies \pi_i^{-1} < \pi_j^{-1} \text{ as integers}\}, \tag{2.1}$$

which is naturally interpreted as a subset of the symmetric group S_n. Note that the identity permutation e always belongs to $\mathcal{L}$. Let P_j be

the natural (induced) subposet of P consisting of elements k such that $j \preceq k$ [Sta97].

We now briefly recall the idea of **promotion** of a linear extension of a poset P. Start with a linear extension $\pi \in \mathcal{L}(P)$ and imagine placing the label π_i^{-1} in P at the location i. By the definition of the linear extension, the labels will be well-ordered. The action of promotion of π will give another linear extension of P as follows:

(1) The process starts with a seed, the label 1. First remove it and replace it by the minimum of all the labels covering it, i, say.
(2) Now look for the minimum of all labels covering i, and replace it, and continue in this way.
(3) This process ends when a label is a "local maximum." Place the label $n+1$ at that point.
(4) Decrease all the labels by 1.

This new linear extension is denoted $\pi\partial$ [Sta09]. We now generalize this to **extended promotion**, whose seed is any of the numbers $1, 2, \ldots, n$. The algorithm is similar to the original one, and we describe it for seed j. Start with the subposet P_j and perform steps 1–3 in a completely analogous fashion. Now decrease all the labels strictly larger than j by 1 in P (not only P_j). Clearly this gives a new linear extension, which we denote $\pi\partial_j$. Note that ∂_n is always the identity.

The extended promotion operator can be expressed in terms of more elementary operators τ_i $(1 \le i < n)$ as shown in [Hai92, MR94, Sta09] and has explicitly been used to count linear extensions in [EHS89]. Let $\pi = \pi_1 \cdots \pi_n \in \mathcal{L}(P)$ be a linear extension of a finite poset P in one-line notation. Then

$$\pi\tau_i = \begin{cases} \pi_1 \cdots \pi_{i-1}\pi_{i+1}\pi_i \cdots \pi_n & \text{if } \pi_i \text{ and } \pi_{i+1} \text{ are not} \\ & \text{comparable in } P, \\ \pi_1 \cdots \pi_n & \text{otherwise.} \end{cases} \tag{2.2}$$

Alternatively, τ_i acts non-trivially on a linear extension if interchanging entries π_i and π_{i+1} yields another linear extension. Then as an operator on $\mathcal{L}(P)$,

$$\partial_j = \tau_j \tau_{j+1} \cdots \tau_{n-1}. \tag{2.3}$$

2.2. **Properties of τ_i and extended promotion.** The operators τ_i are involutions ($\tau_i^2 = 1$) and commute ($\tau_i\tau_j = \tau_j\tau_i$ when $|i - j| > 1$). Unlike the generators for the symmetric group, the τ_i do not always satisfy the braid relation $\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}$. They do, however, satisfy $(\tau_i\tau_{i+1})^6 = 1$ [Sta09].

Proposition 2.1. *Let P be a poset on $[n]$. The braid relations*

$$\pi\tau_j\tau_{j+1}\tau_j = \pi\tau_{j+1}\tau_j\tau_{j+1}$$

hold for all $1 \leq j < n-1$ and all $\pi \in \mathcal{L}(P)$ if and only if P is a union of disjoint chains.

The proof is an easy case-by-case check. Since we do not use this result, we omit the proof.

It will also be useful to express the operators τ_i in terms of the generalized promotion operator.

Lemma 2.2. *For all $1 \leq j \leq n-1$, each operator τ_j can be expressed as a product of promotion operators.*

Proof. We prove the claim by induction on j, starting with the case that $j = n-1$ and decreasing until we reach the case that $j = 1$. When $j = n-1$, the claim is obvious since $\tau_{n-1} = \partial_{n-1}$. For $j < n-1$, we observe that

$$\begin{aligned} \tau_j &= \tau_j\tau_{j+1}\cdots\tau_{n-1}\tau_{n-1}\cdots\tau_{j+2}\tau_{j+1} \\ &= \partial_j\tau_{n-1}\cdots\tau_{j+2}\tau_{j+1}. \end{aligned}$$

By our inductive hypothesis, each of $\tau_{j+1},\dots,\tau_{n-1}$ can be expressed as a product of promotion operators, and hence so too can τ_j. $\square$

3. Various Markov chains

We now consider various continuous time Markov chains related to the extended promotion operator. For completeness, we briefly review the part of the theory relevant to us.

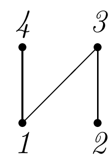
Fix a finite poset P of size n. The operators $\{\tau_i \mid 1 \leq i < n\}$ (resp. $\{\partial_i \mid 1 \leq i \leq n\}$), define a directed graph on the set of linear extensions $\mathcal{L}(P)$. The vertices of the graph are the elements in $\mathcal{L}(P)$ and there is an edge from π to π' if $\pi' = \pi\tau_i$ (resp. $\pi' = \pi\partial_i$). We can now consider continuous time random walks on this graph with weights given by formal indeterminates $x_1,\dots,x_n$. In each case we give two ways to assign the edge weights, see Sections 3.1–3.4. An edge with weight x_i is traversed with that rate. A priori, the x_i's must be positive real numbers for this to make sense according to the standard techniques of Markov chains. However, the ideas seem to work in much greater generality and one can think of this as an "analytic continuation."

The continuous time Markov chain is defined by the **transition matrix** or **generator**, whose coordinates in this case are labeled by elements of $\mathcal{L}(P)$, where we take the convention that the (π',π) entry counts the sum of the weights of edges joining $\pi \to \pi'$ if $\pi \neq \pi'$. On

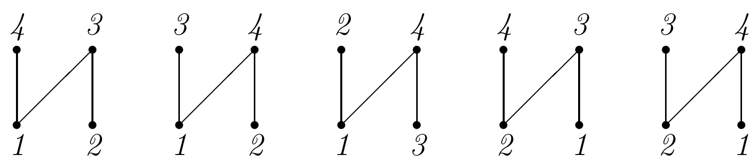
the diagonal, we have the negative sum of the weights of edges leaving π at (π,π) not counting loops. This ensures that column sums are zero and consequently, zero is an eigenvalue with row (left-) eigenvector being the vector $(1,1,1,1,1)$. A Markov chain is said to be **irreducible** if the associated digraph is strongly connected. For such chains, the Perron-Frobenius theorem says that there is a unique eigenvector with eigenvalue zero. The entries of the corresponding column (right-) eigenvector are positive, and suitably normalized, give the **stationary distribution**. For more on the theory of finite state Markov chains, refer to [LPW09].

We set up a running example that will be used for each case.

Example 3.1. *Define P by its covering relations $\{(1,3),(1,4),(2,3)\}$, so that its Hasse diagram is as shown below:*



Then the elements of $\mathcal{L}(P) = \{1234, 1243, 1423, 2134, 2143\}$ are represented by the following diagrams respectively:



3.1. **Uniform transposition graph.** The vertices of the **uniform transposition graph** are the elements in $\mathcal{L}(P)$ and there is an edge between π and π' if and only if $\pi' = \pi\tau_j$ for some $j \in [n-1]$. This edge is assigned the symbolic weight x_j.

Example 3.2. *Consider the poset and linear extensions of Example 3.1. The uniform transposition graph is illustrated in Figure 1. The transition matrix, with the lexicographically ordered basis, is given by*

$$\begin{pmatrix} -x_1-x_3 & x_3 & 0 & x_1 & 0 \\ x_3 & -x_1-x_2-x_3 & x_2 & 0 & x_1 \\ 0 & x_2 & -x_2 & 0 & 0 \\ x_1 & 0 & 0 & -x_1-x_3 & x_3 \\ 0 & x_1 & 0 & x_3 & -x_1-x_3 \end{pmatrix}.$$

By definition the weight x_4 never appears since τ_4 does not exist for $n=4$. Also, by construction the column sums of the transition matrix

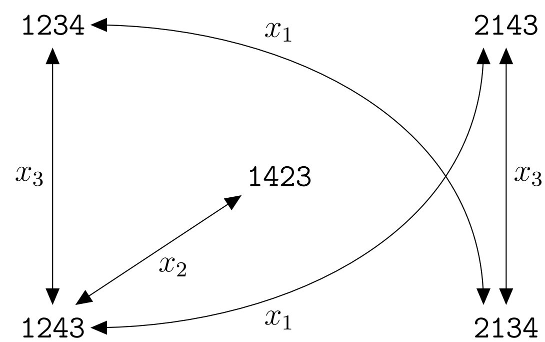


FIGURE 1. Uniform transposition graph for Example 3.1

are zero. Note that in this example the row sums are also zero, which means that the stationary state of this Markov chain is uniform. We will prove this in general in Theorem 4.4.

3.2. **Transposition graph.** The **transposition graph** is defined in the same way as the uniform transposition graph, except that the edges are given the symbolic weight x_{π_j} whenever τ_j takes $\pi \to \pi'$.

Example 3.3. *The transposition graph for the poset in Example 3.1 is illustrated in Figure 2. The transition matrix is given by*

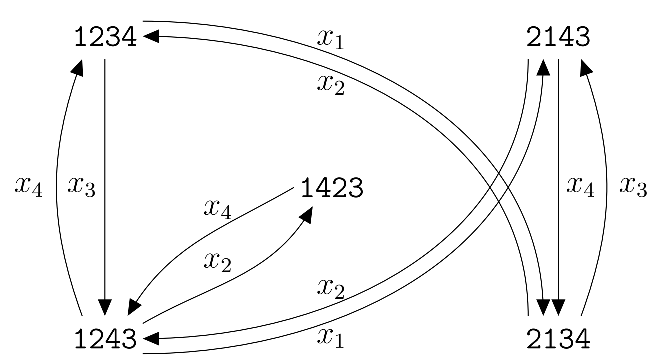


FIGURE 2. Transposition graph for Example 3.1

$$\begin{pmatrix} -x_1 - x_3 & x_4 & 0 & x_2 & 0 \\ x_3 & -x_1 - x_2 - x_4 & x_4 & 0 & x_2 \\ 0 & x_2 & -x_4 & 0 & 0 \\ x_1 & 0 & 0 & -x_2 - x_3 & x_4 \\ 0 & x_1 & 0 & x_3 & -x_2 - x_4 \end{pmatrix}. \tag{3.1}$$

Again, by definition the column sums are zero, but the row sums are not zero in this example. In fact, the stationary distribution (with

eigenvalue 0) is given by the eigenvector

$$\left(1,\quad \frac{x_3}{x_4},\quad \frac{x_2x_3}{{x_4}^2},\quad \frac{x_1}{x_2},\quad \frac{x_1x_3}{x_2x_4}\right)^T \quad . \tag{3.2}$$

We give a closed form expression for the weights of the stationary distribution in the general case in Theorem 4.7.

3.3. **Uniform promotion graph.** The vertices of the **uniform promotion graph** are labeled by elements of $\mathcal{L}(P)$ and there is an edge between π and π' if and only if $\pi' = \pi\partial_j$ for some $j \in [n]$. In this case, the edge is given the symbolic weight x_j.

Example 3.4. *The uniform promotion graph for the poset in Example 3.1 is illustrated in Figure 3. The transition matrix, with the lexicographically ordered basis, is given by*

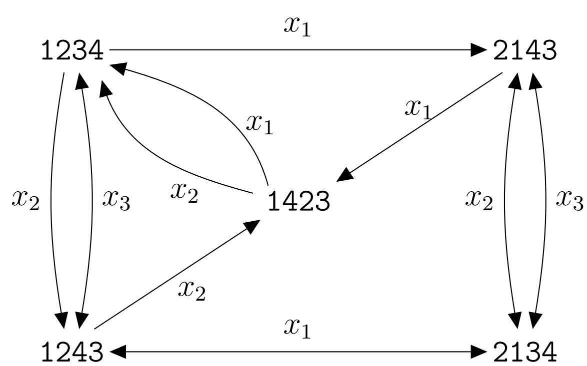


FIGURE 3. Uniform promotion graph for Example 3.1

$$\begin{pmatrix} -x_1-x_2-x_3 & x_3 & x_1+x_2 & 0 & 0 \\ x_2+x_3 & -x_1-x_2-x_3 & 0 & x_1 & 0 \\ 0 & x_2 & -x_1-x_2 & 0 & x_1 \\ 0 & x_1 & 0 & -x_1-x_2-x_3 & x_2+x_3 \\ x_1 & 0 & 0 & x_2+x_3 & -x_1-x_2-x_3 \end{pmatrix}$$

Note that as in Example 3.2 the row sums are zero, so that the stationary state of this Markov chain is uniform. We prove this for general finite posets in Theorem 4.3.

Observe also that x_4 does not occur in the above transition matrix. This is because the action of ∂_4 (or in general ∂_n) maps every linear extension to itself resulting in a loop.

3.4. **Promotion graph.** The **promotion graph** is defined in the same fashion as the uniform promotion graph with the exception that the edge between π and π' when $\pi' = \pi\partial_j$ is given the weight x_{π_j}.

Example 3.5. *The promotion graph for the poset of Example 3.1 is illustrated in Figure 4. Although it might appear that there are many more edges here than in Figure 3, this is not the case. The transition*

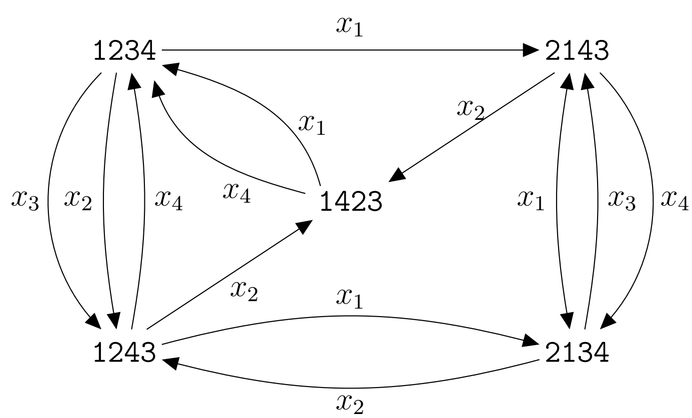


FIGURE 4. Promotion graph for Example 3.1

matrix this time is given by

$$\begin{pmatrix} -x_1-x_2-x_3 & x_4 & x_1+x_4 & 0 & 0 \\ x_2+x_3 & -x_1-x_2-x_4 & 0 & x_2 & 0 \\ 0 & x_2 & -x_1-x_4 & 0 & x_2 \\ 0 & x_1 & 0 & -x_1-x_2-x_3 & x_1+x_4 \\ x_1 & 0 & 0 & x_1+x_3 & -x_1-x_2-x_4 \end{pmatrix}$$

Notice that row sums are no longer zero. The stationary distribution, as a vector written in row notation is

$$\left(1, \quad \frac{x_1+x_2+x_3}{x_1+x_2+x_4}, \quad \frac{(x_1+x_2)(x_1+x_2+x_3)}{(x_1+x_2)(x_1+x_2+x_4)}, \quad \frac{x_1}{x_2}, \quad \frac{x_1(x_1+x_2+x_3)}{x_2(x_1+x_2+x_4)}\right)^T .$$

Again, we will give a general such result in Theorem 4.5.

In Appendix A, implementations of these Markov chains in `Sage` and `Maple` are discussed.

4. Properties of the various Markov chains

In Section 4.1 we prove that the Markov chains defined in Section 3 are all irreducible. This is used in Section 4.2 to conclude that their

stationary state is unique and either uniform or given by an explicit product formula in their weights.

Throughout this section we fix a poset P of size n and let $\mathcal{L} := \mathcal{L}(P)$ be the set of its linear extensions.

4.1. **Irreducibility.** We now show that the four graphs of Section 3 are all **strongly connected**.

Proposition 4.1. *Consider the digraph G whose vertices are labeled by elements of $\mathcal{L}$ and whose edges are given as follows: for $\pi, \pi' \in \mathcal{L}$, there is an edge between π and π' in G if and only if $\pi' = \pi\partial_j$ (resp. $\pi' = \pi\tau_j$) for some $j \in [n]$ (resp. $j \in [n-1]$). Then G is strongly connected.*

Proof. We begin by showing the statement for the generalized promotion operators ∂_j. From an easy generalization of [Sta09], we see that extended promotion, given by ∂_j, is a bijection for any j. Therefore, every element of $\mathcal{L}$ has exactly one such edge pointing in and one such edge pointing out. Moreover, ∂_j has finite order, so that $\pi\partial_j^k = \pi$ for some k. In other words, the action of ∂_j splits $\mathcal{L}$ into disjoint cycles. In particular, $\pi\partial_n = \pi$ for all π so that it decomposes $\mathcal{L}$ into cycles of size 1.

It suffices to show that there is a directed path from any π to the identity e. We prove this by induction on n. The case of the poset with a single element is vacuous. Suppose the statement is true for every poset of size $n-1$. We have two cases. First, suppose $\pi_1^{-1} = 1$. In this case $\partial_2, \ldots, \partial_n$ act on $\mathcal{L}$ in exactly the same way as $\partial_1, \ldots, \partial_{n-1}$ on $\mathcal{L}'$, the set of linear extensions of P', the poset obtained from P by removing 1. Then the directed path exists by the induction assumption.

Instead suppose $\pi_1^{-1} = j$ and $\pi_k^{-1} = 1$, for $j, k > 1$. In other words, the label j is at position 1 and label 1 is at position k of P. Since j is at the position of a minimal element in P, it does not belong to the upper set of 1 (that is $j \not\succeq 1$ in the relabeled poset). Thus, the only effect on j of applying ∂_1 is to reduce it by 1, i.e., if $\pi' = \pi\partial_1$, then $\pi_1'^{-1} = j-1$. Continuing this way, we can get to the previous case by the action of ∂_1^{j-1} on π.

The statement for the τ_j now follows from Lemma 2.2. □

Corollary 4.2. *Assuming that the edge weights are strictly positive, all Markov chains of Section 3 are irreducible and hence their stationary state is unique.*

Proof. Since the underlying graph of all four Markov chains of Section 3 is strongly connected by Proposition 4.1, the statement follows from [LPW09, Theorem 20.1]. □

4.2. **Stationary states.** In this section we prove properties of the stationary state of the various continuous time Markov chains defined in Section 3, assuming that all x_i's are strictly positive.

Theorem 4.3. *The continuous time Markov chain according to the uniform promotion graph has the uniform stationary distribution, that is, each linear extension is equally likely to occur.*

Proof. Stanley showed [Sta09] that the promotion operator has finite order, that is $\partial^k = \mathrm{id}$ for some k. The same arguments go through for the extended promotion operators ∂_j. Therefore at each vertex $\pi \in \mathcal{L}(P)$, there is an incoming and outgoing edge corresponding to ∂_j for each $j \in [n]$. For the uniform promotion graph, an edge for ∂_j is assigned weight x_j, and hence the row sum of the transition matrix is zero, which proves the result. Equivalently, the all ones vector is the required eigenvector. □

Theorem 4.4. *The continuous time Markov chain according to the uniform transposition graph has the uniform stationary distribution.*

Proof. Since each τ_j is an involution, every incoming edge with weight x_j has an outgoing edge with the same weight. Another way of saying the same thing is that the transition matrix is symmetric. By definition, the transition matrix is constructed so that column sums are zero. Therefore, row sums are also zero. □

We now turn to the promotion and transposition graphs of Section 3. In this case we find nice product formulas for the stationary weights.

Theorem 4.5. *The stationary state weight $w(\pi)$ of the linear extension $\pi \in \mathcal{L}(P)$ for the continuous time Markov chain for the promotion graph is given by*

$$w(\pi) = \prod_{i=1}^{n} \frac{x_1 + \cdots + x_i}{x_{\pi_1} + \cdots + x_{\pi_i}} \,, \tag{4.1}$$

assuming $w(e) = 1$.

Remark 4.6. *The entries of w do not, in general, sum to one. Therefore this is not a true probability distribution, but this is easily remedied by a multiplicative constant Z_P depending only on the poset P.*

Proof of Theorem 4.5. We prove the theorem by induction on n. By Remark 4.6, it suffices to prove the result for any normalization of $w(\pi)$. In fact, for our purposes it is most convenient to use the normalization

$$w(\pi) = \prod_{i=1}^{n} \frac{1}{x_{\pi_1} + \cdots + x_{\pi_i}}. \tag{4.2}$$

To prove (4.2), we need to show that it satisfies the master equation

$$w(\pi)\left(\sum_{i=1}^{n} x_{\pi_i}\right) = \sum_{\substack{j=1 \\ \pi'=\pi\tau_{n-1}\cdots\tau_j}}^{n} x_{\pi'_j} w(\pi'). \tag{4.3}$$

The left-hand side is the contribution of the outgoing edges, whereas the right-hand side gives the weights of the incoming edges of vertex π.

Singling out the term $j = n$ and setting $\tilde{\pi} := \pi\tau_{n-1}$, the right-hand side of (4.3) becomes

$$\begin{aligned} x_{\pi_n} w(\pi) + \sum_{\substack{j=1 \\ \pi'=\tilde{\pi}\tau_{n-2}\cdots\tau_j}}^{n-1} x_{\pi'_j} w(\pi') & \\ &= x_{\pi_n} w(\pi) + w(\tilde{\pi})(x_{\tilde{\pi}_1} + \cdots + x_{\tilde{\pi}_{n-1}}), \end{aligned}$$

where the second line follows by induction since $\tilde{\pi}$ can be interpreted as a permutation of size $n-1$ (disregarding the n-th entry), and hence (4.3) applies for $n-1$.

We now distinguish two cases: either τ_{n-1} acts trivially on π or not. In the first case, set $\tilde{\pi} = \pi$ and we immediately obtain the left-hand side of (4.3). In the second case, observe that $w(\pi)$ as in (4.2) satisfies the following recursion if τ_j acts non-trivially

$$w(\pi\tau_j) = \frac{x_{\pi_1} + \cdots + x_{\pi_j}}{x_{\pi_1} + \cdots + x_{\pi_{j-1}} + x_{\pi_{j+1}}} w(\pi).$$

Using this for $j = n-1$ and $x_{\tilde{\pi}_1} + \cdots + x_{\tilde{\pi}_{n-1}} = x_{\pi_1} + \cdots + x_{\pi_{n-2}} + x_{\pi_n}$ yields the left-hand side of (4.3). □

When P is the n-antichain, then $\mathcal{L} = S_n$. In this case, the probability distribution of Theorem 4.5 has been studied in the past by Hendricks [Hen72, Hen73] and is known as the **Tsetlin library** [Tse63], which we now describe. Suppose that a library consists of n books $b_1, \ldots, b_n$ on a single shelf. Assume that only one book is picked at a time and is returned before the next book is picked up. The book b_i is picked with probability x_i and placed at the end of the shelf. This is clearly a Markov chain. Its stationary distribution is a special case of Theorem 4.5. In this case, Z_P of Remark 4.6 also has a nice product formula, leading to the probability distribution,

$$w(\pi) = \prod_{i=1}^{n} \frac{x_{\pi_i}}{x_{\pi_1} + \cdots + x_{\pi_i}}. \tag{4.4}$$

Letac [Let78] considered generalizations of the Tsetlin library to rooted trees (meaning that each element in P besides the root has precisely one successor). Our results hold for any finite poset P.

Theorem 4.7. *The stationary state weight $w(\pi)$ of the linear extension $\pi \in \mathcal{L}(P)$ of the transposition graph is given by*

$$w(\pi) = \prod_{i=1}^{n} x_{\pi_i}^{i-\pi_i}\,, \tag{4.5}$$

assuming $w(e) = 1$.

Proof. To prove the above result, we need to show that it satisfies the master equation

$$w(\pi)\Big(\sum_{i=1}^{n} x_{\pi_i}\Big) = \sum_{j=1}^{n} x_{\pi_j^{(j)}} w(\pi^{(j)}), \tag{4.6}$$

where $\pi^{(j)} = \pi\tau_j$. Let us compare $\pi^{(j)}$ and π. By definition, they differ at the positions j and $j+1$ at most. Either $\pi^{(j)} = \pi$, or $\pi_j^{(j)} = \pi_{j+1}$ and $\pi_{j+1}^{(j)} = \pi_j$. In the former case, we get a contribution to the right hand side of (4.6) of $x_{\pi_j} w(\pi)$, whereas in the latter, $x_{\pi_{j+1}} w(\pi^{(j)})$. But note that in the latter case by (4.5)

$$\frac{w(\pi^{(j)})}{w(\pi)} = \frac{x_{\pi_{j+1}}^{j-\pi_{j+1}} x_{\pi_j}^{j+1-\pi_j}}{x_{\pi_j}^{j-\pi_j} x_{\pi_{j+1}}^{j+1-\pi_{j+1}}} = \frac{x_{\pi_j}}{x_{\pi_{j+1}}},$$

and the contribution is again $x_{\pi_j} w(\pi)$. Thus the j-th term on the right matches that on the left, and this completes the proof. □

5. Partition functions and eigenvalues for rooted forests

For a certain class of posets, we are able to give an explicit formula for the probability distribution for the promotion graph. Note that this involves computing the partition function Z_P (see Remark 4.6). We can also specify all eigenvalues and their multiplicities of the transition matrix explicitly.

5.1. **Main results.** Before we can state the main theorems of this section, we need to make a couple of definitions. A **rooted tree** is a connected poset, where each node has at most one successor. Note that a rooted tree has a unique largest element. A **rooted forest** is a union of rooted trees. A **lower set** (resp. **upper set**) S in a poset is a subset of the nodes such that if $x \in S$ and $y \preceq x$ (resp. $y \succeq x$), then also $y \in S$. We first give the formula for the partition function.

Theorem 5.1. *Let P be a rooted forest of size n and let $x_{\preceq i} = \sum_{j \preceq i} x_j$. The partition function for the promotion graph is given by*

$$Z_P = \prod_{i=1}^{n} \frac{x_{\preceq i}}{x_1 + \cdots + x_i}. \tag{5.1}$$

Proof. We need to show that $w'(\pi) := Z_P w(\pi)$ with $w(\pi)$ given by (4.1) satisfies

$$\sum_{\pi \in \mathcal{L}(P)} w'(\pi) = 1.$$

We shall do so by induction on n. Assume that the formula is true for all rooted forests of size $n-1$. The main idea is that the last entry of π in one-line notation has to be a maximal element of one of the trees in the poset. Let $P = T_1 \cup T_2 \cup \cdots \cup T_k$, where each T_i is a tree. Moreover, let $\hat{T}_i$ denote the maximal element of T_i. Then

$$\sum_{\pi \in \mathcal{L}(P)} w'(\pi) = \sum_{i=1}^{k} \sum_{\sigma \in P \setminus \{\hat{T}_i\}} w'(\sigma \hat{T}_i) .$$

Using (4.1) and (5.1)

$$w'(\sigma \hat{T}_i) = w'(\sigma) \frac{x_{\preceq \hat{T}_i}}{x_1 + \cdots + x_n},$$

which leads to

$$\sum_{\pi \in \mathcal{L}(P)} w'(\pi) = \sum_{i=1}^{k} \frac{x_{\preceq \hat{T}_i}}{x_1 + \cdots + x_n} \sum_{\sigma \in P \setminus \{\hat{T}_i\}} w'(\sigma).$$

By the induction assumption, the rightmost sum is 1, and an easy simplification leads to the desired result. $\square$

Let L be a finite poset with smallest element $\hat{0}$ and largest element $\hat{1}$. Following [Bro00, Appendix C], one may associate to each element $x \in L$ a **derangement number** d_x defined as

$$d_x = \sum_{y \succeq x} \mu(x, y) f([y, \hat{1}]) , \tag{5.2}$$

where $\mu(x, y)$ is the Möbius function for the interval $[x, y] := \{z \in L \mid x \preceq z \preceq y\}$ [Sta97, Section 3.7] and $f([y, \hat{1}])$ is the number of maximal chains in the interval $[y, \hat{1}]$.

A permutation is a **derangement** if it does not have any fixed points. A linear extension π is called a **poset derangement** if it is a derangement when considered as a permutation. Let $\mathfrak{d}_P$ be the number of poset derangements of the poset P.

A **lattice** L is a poset in which any two elements have a unique supremum (also called join) and a unique infimum (also called meet). For $x, y \in L$ the join is denoted by $x \vee y$, whereas the meet is $x \wedge y$. For an **upper semi-lattice** we only require the existence of a unique supremum of any two elements.

Theorem 5.2. *Let P be a rooted forest of size n, M the transition matrix of the promotion graph of Section 3.4, and $\overline{M} = M + (x_1 + x_2 + \cdots + x_n)\mathbb{1}$. Then*

$$\det(\overline{M} - \lambda \mathbb{1}) = \prod_{\substack{S \subseteq [n] \\ S \text{ upper set in } P}} (\lambda - x_S)^{d_S},$$

where $x_S = \sum_{i \in S} x_i$ and d_S is the derangement number in the lattice L (by inclusion) of upper sets in P. In other words, for each subset $S \subseteq [n]$, which is an upper set in P, there is an eigenvalue x_S with multiplicity d_S.

Remark 5.3. *It follows directly from Theorem 5.2, that the matrix M has eigenvalues $-x_S$, where S is a lower set in P, with multiplicity $d_{P \setminus S}$ in the lattice L of upper sets in P.*

The proof of Theorem 5.2 will be given in Section 6. As we will see in Lemma 6.5, the action of the operators in the promotion graph of Section 3.4 for rooted forests have a Tsetlin library type interpretation of moving books to the end of a stack (up to reordering).

When P is a union of chains, which is a special case of rooted forests, we can express the eigenvalue multiplicities directly in terms of the number of poset derangements.

Theorem 5.4. *Let $P = [n_1] + [n_2] + \cdots + [n_k]$ be a union of chains of size n whose elements are labeled consecutively within chains. Then*

$$\det(M - \lambda \mathbb{1}) = \prod_{\substack{S \subseteq [n] \\ S \text{ lower set in } P}} (\lambda + x_S)^{\mathfrak{d}_S},$$

where $\mathfrak{d}_\emptyset = 1$.

The proof of Theorem 5.4 is given in Section 5.2.

Corollary 5.5. *For P a union of chains, we have the identity*

$$|\mathcal{L}(P)| = \sum_{\substack{S \subseteq [n] \\ S \text{ upper set in } P}} d_S = \sum_{\substack{S \subseteq [n] \\ S \text{ lower set in } P}} \mathfrak{d}_S \,. \tag{5.3}$$

Note that the antichain is a special case of a rooted forest and in particular a union of chains. In this case the Markov chain is the Tsetlin library and all subsets of $[n]$ are upper (and lower) sets. Hence Theorem 5.2 specializes to the results of Donnelly [Don91], Kapoor and Reingold [KR91], and Phatarford [Pha91] for the Tsetlin library.

The case of unions of chains, which are consecutively labeled, can be interpreted as looking at a parabolic subgroup of S_n. If there are k chains of lengths n_i for $1 \le i \le k$, then the parabolic subgroup is $S_{n_1} \times \cdots \times S_{n_k}$. In the realm of the Tsetlin library, there are n_i books of the same color. The Markov chain consists of taking a book at random and placing it at the end of the stack.

5.2. **Proof of Theorem 5.4.** We deduce Theorem 5.4 from Theorem 5.2. By Remark 5.3, the matrix M has eigenvalues indexed by lower sets S with multiplicity $d_{P\setminus S}$. We need to show that $\mathfrak{d}_S = d_{P\setminus S}$.

Let P be a union of chains and L the lattice of upper sets of P. The Möbius function of P is the product of the Möbius functions of each chain. This implies that the only upper sets of P with a nonzero entry of the Möbius function are the ones with unions of the top element in each chain.

Since upper sets of unions of chains are again unions of chains, it suffices to consider $d_\emptyset$ for P as d_S can be viewed as $d_\emptyset$ for $P \setminus S$. By (5.2) we have

$$d_\emptyset = \sum_S \mu(\emptyset, S) f([S, \hat{1}]) \ ,$$

where the sum is over all upper sets of P containing only top elements in each chain. Recall that $f([S, \hat{1}])$ is the number of chains from S to $\hat{1}$ in L. By inclusion-exclusion, the claim that $d_\emptyset = \mathfrak{d}_P$ is the number of poset derangements of P, that is the number of linear extensions of P without fixed points, follows from the next lemma.

Lemma 5.6. *Let $P = [n_1]+[n_2]+\cdots+[n_k]$. Fix $I \subseteq [k]$ and let $S \subseteq P$ be the upper set containing the top element of the ith chain of P for all $i \in I$. Then $f([S, \hat{1}])$ is equal to the number of linear extensions of P that fix at least one element of the ith chain of P for all $i \in I$.*

Proof. Let $n = n_1 + n_2 + \cdots + n_k$ denote the number of elements in P. Let $N_1 = 0$ and define $N_i = n_1 + \cdots + n_{i-1}$ for all $2 \le i \le k$. We label the elements of P consecutively so that $N_i + 1, N_i + 2, \ldots, N_{i+1}$ label the elements of the ith chain of P for all $1 \le i \le k$.

The linear extensions of P are in bijection with words w of length n in the alphabet $\mathcal{E} := \{e_1, e_2, \ldots, e_k\}$ with n_i instances of each letter e_i. Indeed, given a linear extension π of P, we associate such a word

w to π by setting $w_j = e_i$ if $\pi_j \in \{N_i + 1, \ldots, N_{i+1}\}$; i.e. if j lies in the ith column of P under the extension π. For the remainder of the proof, we will identify a linear extension π (and properties of π) with its corresponding word w. We also view e_i as standard basis vectors in $\mathbb{Z}^k$.

For any $1 \leq i \leq k$ and $1 \leq j \leq n_i$, the element $N_i + j$ is fixed by w if and only if w satisfies the following two conditions:

- $w_{N_i+j} = e_i$ (i.e. w sends $N_i + j$ to the ith column of P) and
- the restriction of w to its first $N_i + j$ letters, which we denote $w|_{[1,\ldots,N_i+j]}$, contains exactly j instances of the letter e_i (i.e. $N_i + j$ is the jth element of the ith column of P under the extension w).

Moreover, it is clear that the set of all $j \in \{1, \ldots, n_i\}$ such that w fixes $N_i + j$ is an interval of the form $[a_i, b_i]$.

With I and S defined as in the statement of the Lemma, let

$$n'_i := \begin{cases} n_i - 1 & \text{if } i \in I, \\ n_i & \text{if } i \notin I. \end{cases}$$

Similarly, define $N'_1 = 0$ and $N'_i = n'_1 + \cdots + n'_{i-1}$ for $i \geq 2$. We see that $f([S, \hat{1}])$ counts the number of words of length $n - |I|$ in the alphabet $\mathcal{E}$ with n'_j instances of each letter e_j. This is because S corresponds to the element δ_I defined by

$$\delta_I(i) = \begin{cases} 1 & \text{if } i \in I, \\ 0 & \text{if } i \notin I, \end{cases}$$

of L. The maximal chains in L from δ_I to $(n_1, n_2, \ldots, n_k)$ are lattice paths in $\mathbb{Z}^k$ with steps in the directions of the standard basis vectors $e_1, e_2, \ldots, e_k$.

Having established this notation, we are ready to prove the main statement of the Lemma. Let $\mathcal{W}$ denote the collection of all words in the alphabet $\mathcal{E}$ of length n with n_j instances of each letter e_j that fix an element of the ith chain of P for all $i \in I$. Let $\mathcal{W}'$ denote the collection of all words of length $n - |I|$ in the alphabet $\mathcal{E}$ with n'_j instances of each letter e_j.

We define a bijection $\varphi : \mathcal{W} \to \mathcal{W}'$ as follows. For each $i \in I$, suppose $w \in \mathcal{W}$ fixes the elements $N_i + a_i, \ldots, N_i + b_i$ from the ith chain of P. We define $\varphi(w)$ to be the word obtained from w by removing the letter e_i in position $w_{N_i+b_i}$ for each $i \in I$. Clearly $\varphi(w)$ has length $n - |I|$ and n'_j instances of each letter e_j.

Conversely, given $w' \in \mathcal{W}'$, let J_i be the set of indices $N'_i + j$ with $0 \leq j \leq n'_i$ such that $w'|_{[1,\ldots,N'_i+j]}$ contains exactly j instances of the

letter e_i. Here we allow $j = 0$ since it is possible that there are no instances of the letter e_i among the first N'_i letters of w'. Again, it is clear that each J_i is an interval of the form $[N'_i + c_i, \ldots, N'_i + d_i]$ and $w'_{N_i+j} = e_i$ for all $j \in [c_i + 1, \ldots, d_i]$, though it is possible that $w'_{N'_i+c_i} \neq e_i$. Thus we define $\varphi^{-1}(w')$ to be the word obtained from w' by inserting the letter e_i after $w'_{N'_i+d_i}$ for all $i \in I$. $\square$

We illustrate the proof of Lemma 5.6 in the following example.

Example 5.7. *Let* $P = [3] + [4] + [2] + [5]$, $I = \{2, 4\}$, *and consider the linear extension*

$$\pi := 1\ 10\ 4\ 8\ \mathbf{5}\ \mathbf{6}\ 2\ 3\ 11\ 9\ 7\ \mathbf{12}\ \mathbf{13}\ \mathbf{14},$$

which corresponds to the word

$$w = e_1e_4e_2|e_3\mathbf{e_2}\mathbf{e_2}e_1|e_1e_4|e_3e_2\mathbf{e_4}\mathbf{e_4}\mathbf{e_4}.$$

Here we have divided the word according to the chains of P. *The fixed points of* π *in the second and fourth chains of* P *are shown in bold, along with their corresponding entries of the word* w. *In this case* $\varphi(w) = e_1e_4e_2e_3e_2e_1e_1e_4e_3e_2e_4e_4$.

Conversely, consider

$$w' = e_2e_1e_4|e_3e_3e_1|e_2e_1|e_2e_4e_4e_4 \in \mathcal{W}'.$$

Again, we have partitioned w' *into blocks of size* n'_i *for each* $i = 1, \ldots, 4$. *In this case,* $J_2 = \{4\}$ *and* $J_4 = \{10, 11, 12\}$, *so* $\varphi^{-1}(w')$ *is the following word, with the inserted letters shown in bold:*

$$\varphi^{-1}(w') = e_1e_1e_4|e_3\mathbf{e_2}e_1e_3|e_2e_1|e_2e_4e_4e_4\mathbf{e_4}.$$

Remark 5.8. *The initial labeling of* P *in the proof of Lemma 5.6 is essential to the proof. For example, let* P *be the poset* $[2] + [2]$ *with two chains, each of length two. Labeling the elements of* P *so that* $1 < 2$ *and* $3 < 4$ *admits two derangements:* 3142 *and* 3412. *On the other hand, labeling the elements of* P *so that* $1 < 4$ *and* $2 < 3$ *only admits one derangement:* 2143. *In either case, the eigenvalue* $-(x_1 + x_2 + x_3 + x_4)$ *of* $\overline{M}$ *has multiplicity* 2.

6. $\mathcal{R}$-trivial monoids

In this section we provide the proof of Theorem 5.2. We first note that in the case of rooted forests the monoid generated by the relabeled promotion operators of the promotion graph is $\mathcal{R}$-trivial (see Sections 6.1 and 6.2). Then we use a generalization of Brown's theory [Bro00] for Markov chains associated to left regular bands (see also [Bid97, BHR99]) to $\mathcal{R}$-trivial monoids. This is in fact a special

case of Steinberg's results [Ste06, Theorems 6.3 and 6.4] for monoids in the pseudovariety **DA** as stated in Section 6.3. The proof of Theorem 5.2 is given in Section 6.4.

6.1. **$\mathcal{R}$-trivial monoids.** A finite **monoid** $\mathcal{M}$ is a finite set with an associative multiplication and an identity element. Green [Gre51] defined several preorders on $\mathcal{M}$. In particular for $x, y \in \mathcal{M}$ right and left order is defined as

$$\begin{aligned} x &\le_{\mathcal{R}} y \quad \text{if } y = xu \text{ for some } u \in \mathcal{M},\\ x &\le_{\mathcal{L}} y \quad \text{if } y = ux \text{ for some } u \in \mathcal{M}. \end{aligned} \tag{6.1}$$

(Note that this is in fact the opposite convention used by Green). This ordering gives rise to equivalence classes ($\mathcal{R}$-classes or $\mathcal{L}$-classes)

$$\begin{aligned} x \,\mathcal{R}\, y &\quad \text{if and only if } x\mathcal{M} = y\mathcal{M},\\ x \,\mathcal{L}\, y &\quad \text{if and only if } \mathcal{M}x = \mathcal{M}y. \end{aligned}$$

The monoid $\mathcal{M}$ is said to be **$\mathcal{R}$-trivial** (resp. **$\mathcal{L}$-trivial**) if all $\mathcal{R}$-classes (resp. $\mathcal{L}$-classes) have cardinality one.

Remark 6.1. *A monoid $\mathcal{M}$ is a left regular band if $x^2 = x$ and $xyx = xy$ for all $x, y \in \mathcal{M}$. It is not hard to check (see also* [BBBS11, Example 2.4]*) that left regular bands are $\mathcal{R}$-trivial.*

Schocker [Sch08] introduced the notion of weakly ordered monoids which is equivalent to the notion of $\mathcal{R}$-triviality [BBBS11, Theorem 2.18] (the proof of which is based on ideas by Steinberg and Thiéry).

Definition 6.2. *A finite monoid $\mathcal{M}$ is said to be **weakly ordered** if there is a finite upper semi-lattice $(L^{\mathcal{M}}, \preceq)$ together with two maps* $\operatorname{supp}, \operatorname{des} : \mathcal{M} \to L^{\mathcal{M}}$ *satisfying the following axioms:*

(1) supp *is a surjective monoid morphism, that is,* $\operatorname{supp}(xy) = \operatorname{supp}(x) \vee \operatorname{supp}(y)$ *for all $x, y \in \mathcal{M}$ and* $\operatorname{supp}(\mathcal{M}) = L^{\mathcal{M}}$.
(2) If $x, y \in \mathcal{M}$ are such that $xy \le_{\mathcal{R}} x$, then $\operatorname{supp}(y) \preceq \operatorname{des}(x)$.
(3) If $x, y \in \mathcal{M}$ are such that $\operatorname{supp}(y) \preceq \operatorname{des}(x)$*, then $xy = x$.*

Theorem 6.3. [BBBS11, Theorem 2.18] *Let $\mathcal{M}$ be a finite monoid. Then $\mathcal{M}$ is weakly ordered if and only if $\mathcal{M}$ is $\mathcal{R}$-trivial.*

If $\mathcal{M}$ is $\mathcal{R}$-trivial, then for each $x \in \mathcal{M}$ there exists an exponent of x such that $x^\omega x = x^\omega$. In particular x^ω is idempotent, that is, $(x^\omega)^2 = x^\omega$.

Given an $\mathcal{R}$-trivial monoid $\mathcal{M}$, one might be interested in finding the underlying semi-lattice $L^{\mathcal{M}}$ and maps $\operatorname{supp}, \operatorname{des}$.

Remark 6.4. *The upper semi-lattice $L^{\mathcal{M}}$ and the maps* $\operatorname{supp}, \operatorname{des}$ *for an $\mathcal{R}$-trivial monoid $\mathcal{M}$ can be constructed as follows:*

(1) $L^{\mathcal{M}}$ *is the set of left ideals* $\mathcal{M}e$ *generated by the idempotents* $e \in \mathcal{M}$*, ordered by reverse inclusion.*
(2) $\operatorname{supp} : \mathcal{M} \to L^{\mathcal{M}}$ *is defined as* $\operatorname{supp}(x) = \mathcal{M}x^{\omega}$*.*
(3) $\operatorname{des} : \mathcal{M} \to L^{\mathcal{M}}$ *is defined as* $\operatorname{des}(x) = \operatorname{supp}(e)$*, where* e *is some maximal element in the set* $\{y \in \mathcal{M} \mid xy = x\}$ *with respect to the preorder* $\leq_{\mathcal{R}}$*.*

The idea of associating a lattice (or semi-lattice) to certain monoids has been used for a long time in the semigroup community [CP61].

6.2. **$\mathcal{R}$-triviality of the promotion monoid.** Now let P be a rooted forest of size n and $\hat{\partial}_i$ for $1 \leq i \leq n$ the operators on $\mathcal{L}(P)$ defined by the promotion graph of Section 3.4. That is, for $\pi, \pi' \in \mathcal{L}(P)$, the operator $\hat{\partial}_i$ maps π to π' if $\pi' = \pi \partial_{\pi_i^{-1}}$. We are interested in the monoid $\mathcal{M}^{\hat{\partial}}$ generated by $\{\hat{\partial}_i \mid 1 \leq i \leq n\}$.

Lemma 6.5. *Let* P *and* $\hat{\partial}_i$ *be as above, and* $\pi \in \mathcal{L}(P)$*. Then* $\pi\hat{\partial}_i$ *is the linear extension in* $\mathcal{L}(P)$ *obtained from* π *by moving the letter* i *to position* n *and reordering all letters* $j \succeq i$*.*

Proof. Suppose $\pi_i^{-1} = k$. Then the letter i is in position k in π. Furthermore by definition $\pi\hat{\partial}_{\pi_i^{-1}} = \pi\hat{\partial}_k = \pi\tau_k\tau_{k+1}\cdots\tau_{n-1}$. Since π is a linear extension of P, all comparable letters are ordered within π. Hence τ_k either tries to switch i with a letter $j \succeq i$ or an incomparable letter j. In the case $j \succeq i$, τ_k acts as the identity. In the other case τ_k switches the elements. In the first (resp. second) case we repeat the argument with i replaced by its unique successor j (resp. i) and τ_k replaced by τ_{k+1} etc.. It is not hard to see that this results in the claim of the lemma. □

Example 6.6. *Let* P *be the union of a chain of length 3 and a chain of length 2, where the first chain is labeled by the elements* $\{1, 2, 3\}$ *and the second chain by* $\{4, 5\}$*. Then* $41235\ \hat{\partial}_1 = 41253$*, which is obtained by moving the letter 1 to the end of the word and then reordering the letters* $\{1, 2, 3\}$*, so that the result is again a linear extension of* P*.*

Let $x \in \mathcal{M}^{\hat{\partial}}$. The image of x is $\operatorname{im}(x) = \{\pi x \mid \pi \in \mathcal{L}(P)\}$. Furthermore, for each $\pi \in \operatorname{im}(x)$, let $\operatorname{fiber}(\pi, x) = \{\pi' \in \mathcal{L}(P) \mid \pi = \pi' x\}$. Let $\operatorname{rfactor}(x)$ be the maximal common right factor of all elements in $\operatorname{im}(x)$, that is, all elements $\pi \in \operatorname{im}(x)$ can be written as $\pi = \pi_1 \cdots \pi_m \operatorname{rfactor}(x)$ and there is no bigger right factor for which this is true. Let us also define the set of entries in the right factor $\operatorname{Rfactor}(x) = \{i \mid i \in \operatorname{rfactor}(x)\}$. Note that since all elements in the

image set of x are linear extensions of P, $\mathrm{Rfactor}(x)$ is an upper set of P.

By Lemma 6.5 linear extensions in $\mathrm{im}(\hat{\partial}_i)$ have as their last letter $\max_P\{j \mid j \succeq i\}$; this maximum is unique since P is a rooted forest. Hence it is clear that $\mathrm{im}(\hat{\partial}_i x) \subseteq \mathrm{im}(x)$ for any $x \in \mathcal{M}^{\hat{\partial}}$ and $1 \le i \le n$. In particular, if $x \le_{\mathcal{L}} y$, that is $y = ux$ for some $u \in \mathcal{M}^{\hat{\partial}}$, then $\mathrm{im}(y) \subseteq \mathrm{im}(x)$. Hence x, y can only be in the same $\mathcal{L}$-class if $\mathrm{im}(x) = \mathrm{im}(y)$.

Fix $x \in \mathcal{M}^{\hat{\partial}}$ and let the set $I_x = \{i_1, \ldots, i_k\}$ be maximal such that $\hat{\partial}_{i_j} x = x$ for $1 \le j \le k$. The following holds.

Lemma 6.7. *If x is an idempotent, then* $\mathrm{Rfactor}(x) = I_x$.

Proof. Recall that the operators $\hat{\partial}_i$ generate $\mathcal{M}^{\hat{\partial}}$. Hence we can write $x = \hat{\partial}_{\alpha_1} \cdots \hat{\partial}_{\alpha_m}$ for some $\alpha_j \in [n]$. The condition $\hat{\partial}_i x = x$ is equivalent to the condition that for every $\pi \in \mathcal{L}(P)$ there is a $\pi' \in \mathcal{L}(P)$ such that $\mathrm{fiber}(\pi, \hat{\partial}_i) \subseteq \mathrm{fiber}(\pi', x)$ (meaning that the fibers of x are coarser than the fibers of $\hat{\partial}_i$) and $\mathrm{im}(\hat{\partial}_i) \cap \mathrm{fiber}(\pi, x) \neq \emptyset$ for all $\pi \in \mathrm{im}(x)$ (meaning that $\mathrm{im}(\hat{\partial}_i x) = \mathrm{im}(x)$). Since $x^2 = x$ is an idempotent, we hence must have $\hat{\partial}_{\alpha_j} x = x$ for all $1 \le j \le m$.

Now let us consider $x\hat{\partial}_{\alpha_j}$. If $\alpha_j \notin \mathrm{Rfactor}(x)$, then by Lemma 6.5 we have $\mathrm{Rfactor}(x) \subsetneq \mathrm{Rfactor}(x\hat{\partial}_{\alpha_j})$ and hence $|\,\mathrm{im}(x\hat{\partial}_{\alpha_j})| < |\,\mathrm{im}(x)|$, which contradicts the fact that $x^2 = x$. Therefore, $\alpha_j \in \mathrm{Rfactor}(x)$.

Now suppose $\hat{\partial}_i x = x$. Then $x = \hat{\partial}_i \hat{\partial}_{\alpha_1} \cdots \hat{\partial}_{\alpha_m}$ and by the same arguments as above $i \in \mathrm{Rfactor}(x)$. Hence $I_x \subseteq \mathrm{Rfactor}(x)$. Conversely, suppose $i \in \mathrm{Rfactor}(x)$. Then $x\hat{\partial}_i$ has the same fibers as x (but possibly a different image set since $\mathrm{rfactor}(x\hat{\partial}_i) = \mathrm{rfactor}(x)\hat{\partial}_i$ which can be different from $\mathrm{rfactor}(x)$). This implies $x\hat{\partial}_i x = x$. Hence considering the expression in terms of generators $x = \hat{\partial}_{\alpha_1} \cdots \hat{\partial}_{\alpha_m} \hat{\partial}_i \hat{\partial}_{\alpha_1} \cdots \hat{\partial}_{\alpha_m}$, the above arguments imply that $\hat{\partial}_i x = x$. This shows that $\mathrm{Rfactor}(x) \subseteq I_x$ and hence $I_x = \mathrm{Rfactor}(x)$. This proves the claim. □

Lemma 6.8. *I_x is an upper set of P for any $x \in \mathcal{M}^{\hat{\partial}}$. More precisely,* $I_x = \mathrm{Rfactor}(e)$ *for some idempotent $e \in \mathcal{M}^{\hat{\partial}}$.*

Proof. For any $x \in \mathcal{M}^{\hat{\partial}}$, $\mathrm{rfactor}(x) \subseteq \mathrm{rfactor}(x^\ell)$ for any integer $\ell > 0$. Also, the fibers of x^ℓ are coarser or equal to the fibers of x. Since the right factors can be of length at most n (the size of P) and $\mathcal{M}^{\hat{\partial}}$ is finite, for ℓ sufficiently large we have $(x^\ell)^2 = x^\ell$, so that x^ℓ is an idempotent. Now take a maximal idempotent e in the $\ge_{\mathcal{R}}$ preorder such that $ex = x$ (when $I_x = \emptyset$ we have $e = \mathbb{1}$) which exists by the previous arguments. Then $I_e = I_x$ which by Lemma 6.7 is also $\mathrm{Rfactor}(e)$. This proves the claim. □

Let M be the transition matrix of the promotion graph of Section 3.4, and $\overline{M} = M + (x_1 + x_2 + \cdots + x_n)\mathbb{1}$. Define $\mathcal{M}$ to be the monoid generated by $\{G_i \mid 1 \leq i \leq n\}$, where G_i is the matrix $\overline{M}$ evaluated at $x_i = 1$ and all other $x_j = 0$. We are now ready to state the main result of this section.

Theorem 6.9. *$\mathcal{M}$ is $\mathcal{R}$-trivial.*

Remark 6.10. *Considering the matrix monoid $\mathcal{M}$ is equivalent to considering the abstract monoid $\mathcal{M}^{\hat{\partial}}$ generated by $\{\hat{\partial}_i \mid 1 \leq i \leq n\}$. Since the operators $\hat{\partial}_i$ act on the left on linear extensions, the monoid $\mathcal{M}^{\hat{\partial}}$ is left trivial instead of right trivial.*

Example 6.11. *Let P be the poset on three elements $\{1, 2, 3\}$, where 2 covers 1 and there are no further relations. The linear extensions of P are $\{123, 132, 312\}$. The monoid $\mathcal{M}$ with $\mathcal{R}$-order, where an edge labeled i means right multiplication by G_i, is depicted in Figure 5. From the picture it is clear that the elements in the monoid are partially ordered. This confirms Theorem 6.9 that the monoid is $\mathcal{R}$-trivial.*

Example 6.12. *Now consider the poset P on three elements $\{1, 2, 3\}$, where 1 is covered by both 2 and 3 with no further relations. The linear extensions of P are $\{123, 132\}$. This poset is not a rooted forest. The corresponding monoid in $\mathcal{R}$-order is depicted in Figure 6. The two elements*

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

are in the same $\mathcal{R}$-class. Hence the monoid is not $\mathcal{R}$-trivial, which is consistent with Theorem 6.9.

Proof of Theorem 6.9. By Theorem 6.3 a monoid is $\mathcal{R}$-trivial if and only if it is weakly ordered. We prove the theorem by explicitly constructing the semi-lattice $L^{\mathcal{M}}$ and maps $\operatorname{supp}, \operatorname{des} : \mathcal{M}^{\hat{\partial}} \to L^{\mathcal{M}}$ of Definition 6.2. In fact, since we work with $\mathcal{M}^{\hat{\partial}}$, we will establish the left version of Definition 6.2 by Remark 6.10.

Recall that for $x \in \mathcal{M}^{\hat{\partial}}$, we defined the set $I_x = \{i_1, \ldots, i_k\}$ to be maximal such that $\hat{\partial}_{i_j} x = x$ for $1 \leq j \leq k$.

Define $\operatorname{des}(x) = I_x$ and $\operatorname{supp}(x) = \operatorname{des}(x^\omega)$. By Lemma 6.7, for idempotents x we have $\operatorname{supp}(x) = \operatorname{des}(x) = I_x = \operatorname{Rfactor}(x)$. Let $L^{\mathcal{M}} = \{\operatorname{Rfactor}(x) \mid x \in \mathcal{M}^{\hat{\partial}}, x^2 = x\}$ which has a natural semi-lattice structure $(L^{\mathcal{M}}, \preceq)$ by inclusion of sets. The join operation is union of sets.

Certainly by Lemma 6.7 and the definition of $L^{\mathcal{M}}$, the map supp is surjective. We want to show that in addition $\operatorname{supp}(xy) = \operatorname{supp}(x) \vee$

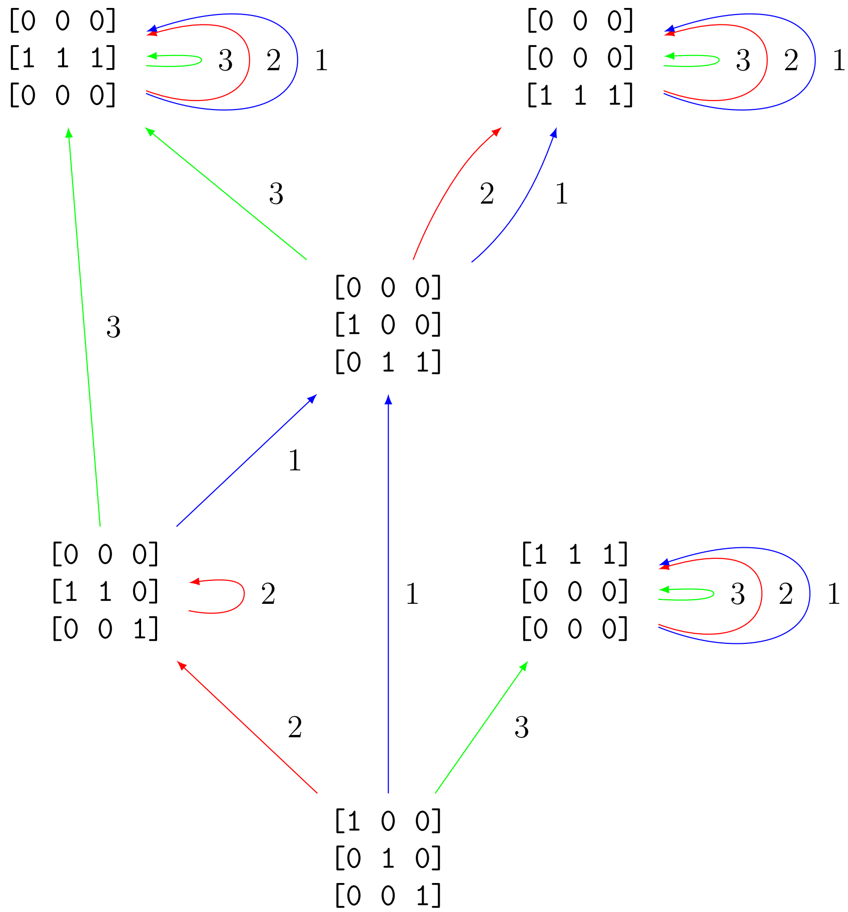


FIGURE 5. Monoid $\mathcal{M}$ in right order for the poset of Example 6.11

supp(y), where $\vee$ is the join in $L^{\mathcal{M}}$. Recall that $\operatorname{supp}(x) = \operatorname{des}(x^\omega) = \operatorname{Rfactor}(x^\omega)$. If $x = \hat{\partial}_{j_1} \cdots \hat{\partial}_{j_m}$ in terms of the generators and $J_x := \{j_1, \ldots, j_m\}$, then by Lemma 6.5 Rfactor(x^ω) contains the upper set of J_x in P plus possibly some more elements that are forced if the upper set of J_x has only one successor in the semi-lattice of upper sets in P. A similar argument holds for y with J_y. Now again by Lemma 6.5, $\operatorname{supp}(xy) = \operatorname{Rfactor}((xy)^\omega)$ contains the elements in the upper set of $J_x \cup J_y$, plus possibly more forced by the same reason as before. Hence $\operatorname{supp}(xy) = \operatorname{supp}(x) \vee \operatorname{supp}(y)$. This shows that Definition 6.2 (1) holds.

Suppose $x, y \in \mathcal{M}^{\hat{\partial}}$ with $yx \leq_{\mathcal{L}} x$. Then there exists a $z \in \mathcal{M}^{\hat{\partial}}$ such that $zyx = x$. Hence $\operatorname{supp}(y) \preceq \operatorname{supp}(zy) \preceq I_x = \operatorname{des}(x)$ by

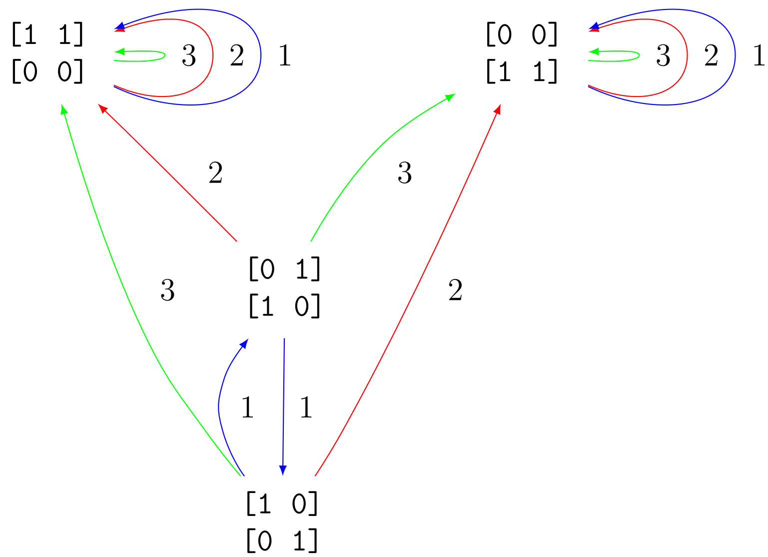


FIGURE 6. Monoid $\mathcal{M}$ in right order for the poset of Example 6.12

Lemmas 6.7 and 6.8. Conversely, if $x, y \in \mathcal{M}^{\hat{\partial}}$ are such that $\operatorname{supp}(y) \preceq \operatorname{des}(x)$, then by the definition of $\operatorname{des}(x)$ we have $\operatorname{supp}(y) \preceq I_x$, which is the list of indices of the left stabilizers of x. By the definition of $\operatorname{supp}(y)$ and the proof of Lemma 6.7, y^ω can be written as a product of $\hat{\partial}_i$ with $i \in \operatorname{supp}(y)$. The same must be true for y. Hence $yx = x$, which shows that the left version of (2) and (3) of Definition 6.2 hold.

In summary, we have shown that $\mathcal{M}^{\hat{\partial}}$ is weakly ordered in $\mathcal{L}$-preorder and hence $\mathcal{L}$-trivial. This implies that $\mathcal{M}$ is $\mathcal{R}$-trivial. □

Remark 6.13. *In the proof of Theorem 6.9 we explicitly constructed the semi-lattice* $L^{\mathcal{M}} = \{\operatorname{Rfactor}(x) \mid x \in \mathcal{M}^{\hat{\partial}}, x^2 = x\}$ *and the maps* $\operatorname{supp}, \operatorname{des} : \mathcal{M}^{\hat{\partial}} \to L^{\mathcal{M}}$ *of Definition 6.2. Here* $\operatorname{des}(x) = I_x$ *is the set of indices* $I_x = \{i_1, \ldots, i_m\}$ *such that* $\hat{\partial}_{i_j} x = x$ *for all* $1 \le j \le m$ *and* $\operatorname{supp}(x) = \operatorname{des}(x^\omega) = I_{x^\omega} = \operatorname{Rfactor}(x^\omega)$.

Example 6.14. *Let P be the poset of Example 6.11. The monoid $\mathcal{M}$ with $\mathcal{R}$-order, where an edge labeled i means right multiplication by G_i, is depicted in Figure 5. The elements* $x = \mathbb{1}, G_2, G_3, G_2G_3, G_1^2$ *are idempotent with* $\operatorname{supp}(x) = \operatorname{des}(x) = \emptyset, 2, 123, 123,$ *123, respectively. The only non-idempotent element is G_1 with* $\operatorname{supp}(G_1) = 123$ *and* $\operatorname{des}(G_1) = \emptyset$. *The semi-lattice $L^{\mathcal{M}}$ is the left lattice in Figure 7. The right graph in Figure 7 is the lattice L of all upper sets of P.*

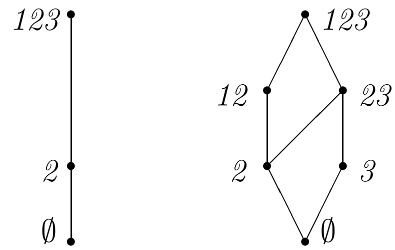


FIGURE 7. The left graph is the lattice $L^{\mathcal{M}}$ of the weakly ordered monoid for the poset in Example 6.14. The right graph is the lattice L of all upper sets of P.

6.3. **Eigenvalues and multiplicities for $\mathcal{R}$-trivial monoids.** Let $\mathcal{M}$ be a finite monoid (for example a left regular band) and $\{w_x\}_{x\in\mathcal{M}}$ a probability distribution on $\mathcal{M}$ with transition matrix for the random walk given by

$$M(c,d) = \sum_{xc=d} w_x \tag{6.2}$$

for $c, d \in \mathcal{C}$, where $\mathcal{C}$ is the set of maximal elements in $\mathcal{M}$ under right order $\geq_{\mathcal{R}}$. The set $\mathcal{C}$ is also called the set of **chambers**.

Recall that by Remark 6.4 we can associate a semi-lattice $L^{\mathcal{M}}$ and functions $\operatorname{supp}, \operatorname{des} : \mathcal{M} \to L^{\mathcal{M}}$ to an $\mathcal{R}$-trivial monoid $\mathcal{M}$. For $X \in L^{\mathcal{M}}$, define c_X to be the number of chambers in $\mathcal{M}_{\geq X}$, that is, the number of $c \in \mathcal{C}$ such that $c \geq_{\mathcal{R}} x$, where $x \in \mathcal{M}$ is any fixed element with $\operatorname{supp}(x) = X$.

Theorem 6.15. *Let $\mathcal{M}$ be a finite $\mathcal{R}$-trivial monoid with transition matrix M as in* (6.2)*. Then M has eigenvalues*

$$\lambda_X = \sum_{\substack{y \\ \operatorname{supp}(y) \preceq X}} w_y \tag{6.3}$$

for each $X \in L^{\mathcal{M}}$ with multiplicity d_X recursively defined by

$$\sum_{Y \succeq X} d_Y = c_X. \tag{6.4}$$

Equivalently,

$$d_X = \sum_{Y \succeq X} \mu(X,Y)\, c_Y, \tag{6.5}$$

where μ is the Möbius function on $L^{\mathcal{M}}$.

Brown [Bro00, Theorem 4, Page 900] proved Theorem 6.15 in the case when $\mathcal{M}$ is a left regular band. Theorem 6.15 is a generalization

to the $\mathcal{R}$-trivial case. It is in fact a special case of a result of Steinberg [Ste06, Theorems 6.3 and 6.4] for monoids in the pseudovariety **DA**. This was further generalized in [Ste08].

6.4. **Proof of Theorem 5.2.** By Theorem 6.9 the promotion monoid $\mathcal{M}$ is $\mathcal{R}$-trivial, hence Theorem 6.15 applies.

Let L be the lattice of upper sets of P and $L^{\mathcal{M}}$ the semi-lattice of Definition 6.2 associated to $\mathcal{R}$-trivial monoids that is used in Theorem 6.15. Recall that for the promotion monoid $L^{\mathcal{M}} = \{\mathrm{Rfactor}(x) \mid x \in \mathcal{M}^{\hat{\partial}}, x^2 = x\}$ by Remark 6.13. Now pick $S \in L$ and let $r = r_1 \ldots r_m$ be any linear extension of $P|_S$ (denoting P restricted to S). By repeated application of Lemma 6.5, it is not hard to see that $x = \hat{\partial}_{r_1} \cdots \hat{\partial}_{r_m}$ is an idempotent since $r_1 \cdots r_m \subseteq \mathrm{rfactor}(x)$ and x only acts on this right factor and fixes it. rfactor(x) is strictly bigger than $r_1 \ldots r_m$ if some further letters beyond $r_1 \ldots r_m$ are forced in the right factors of the elements in the image set. This can only happen if there is only one successor S' of S in the lattice L. In this case the element in $S' \setminus S$ is forced as the letter to the left of $r_1 \ldots r_m$ and is hence part of rfactor(x).

Recall that $f([S, \hat{1}])$ is the number of maximal chains from S to the maximal element $\hat{1}$ in L. Since L is the lattice of upper sets of P, this is precisely the number of linear extensions of $P|_{P\setminus S}$. If $S \in L$ has only one successor S', then $f([S, \hat{1}]) = f([S', \hat{1}])$. Equation (5.2) is equivalent to

$$f([S, \hat{1}]) = \sum_{T \succeq S} d_T$$

(see [Bro00, Appendix C] for more details). Hence $f([S, \hat{1}]) = f([S', \hat{1}])$ implies that $d_S = 0$ in the case when S has only one successor S'.

Now suppose $S \in L^{\mathcal{M}}$ is an element of the smaller semi-lattice. Recall that c_S of Theorem 6.15 is the number of maximal elements in $x \in \mathcal{M}^{\hat{\partial}}$ with $x \geq_{\mathcal{R}} s$ for some s with $\mathrm{supp}(s) = S$. In $\mathcal{M}$ the maximal elements in $\mathcal{R}$-order (or equivalently in $\mathcal{M}^{\hat{\partial}}$ in $\mathcal{L}$-order) form the chamber $\mathcal{C}$ (resp. $\mathcal{C}^{\hat{\partial}}$) and are naturally indexed by the linear extensions in $\mathcal{L}(P)$. Namely, given $\pi = \pi_1 \ldots \pi_n \in \mathcal{L}(P)$ the element $x = \hat{\partial}_{\pi_1} \cdots \hat{\partial}_{\pi_n}$ is idempotent, maximal in $\mathcal{L}$-order and has as image set $\{\pi\}$. Conversely, given a maximal element x in $\mathcal{L}$-order it must have $\mathrm{rfactor}(x) \in \mathcal{L}(P)$. Given $s \in \mathcal{M}^{\hat{\partial}}$ with $\mathrm{supp}(s) = S$, only those maximal elements $x \in \mathcal{M}^{\hat{\partial}}$ associated to $\pi \in \mathrm{im}(s)$ are bigger than s. Hence for $S \in L^{\mathcal{M}}$ we have $c_S = f([S, \hat{1}])$.

The above arguments show that instead of $L^{\mathcal{M}}$ one can also work with the lattice L of upper sets since any $S \in L$ but $S \notin L^{\mathcal{M}}$ comes with multiplicity $d_S = 0$ and otherwise the multiplicities agree.

The promotion Markov chain assigns a weight x_i for a transition from π to π' for $\pi, \pi' \in \mathcal{L}(P)$ if $\pi' = \pi\hat{\partial}_i$. Recall that elements in the chamber $\mathcal{C}^{\hat{\partial}}$ are naturally associated with linear extensions. Let $x, x' \in \mathcal{C}^{\hat{\partial}}$ be associated to π, π', respectively. That is, $\pi = \tau x$ and $\pi' = \tau x'$ for all $\tau \in \mathcal{L}(P)$. Then $x' = x\hat{\partial}_i$ since $\tau(x\hat{\partial}_i) = (\tau x)\hat{\partial}_i = \pi\hat{\partial}_i = \pi'$ for all $\tau \in \mathcal{L}(P)$. Equivalently in the monoid $\mathcal{M}$ we would have $X' = G_i X$ for $X, X' \in \mathcal{C}$. Hence comparing with (6.2), setting the probability variables to $w_{G_i} = x_i$ and $w_X = 0$ for all other $X \in \mathcal{M}$, Theorem 6.15 implies Theorem 5.2.

Example 6.16. *Figure 7 shows the lattice $L^{\mathcal{M}}$ on the left and the lattice L of upper sets of P on the right, for the monoid displayed in Figure 5. The elements $2, 23, 12$ in L have only one successor and hence do not appear in $L^{\mathcal{M}}$.*

7. Outlook

Two of our Markov chains, the uniform promotion graph and the uniform transposition graph, are irreducible and have the uniform distribution as their stationary distributions. Moreover, the former is irreversible and has the advantage of having tunable parameters $x_1, \ldots, x_n$ whose only constraint is that they sum to 1. Because of the irreversibility property, it is plausible that the mixing times for this Markov chain is smaller than the ones considered by Bubley and Dyer [BD99]. Hence the uniform promotion graph could have possible applications for uniformly sampling linear extensions of a large poset. This is certainly deserving of further study.

It would also be interesting to extend the results of Brown and Diaconis [BD98] (see also [AD10]) on rates of convergences to the Markov chains in this paper. For the Markov chains corresponding to $\mathcal{R}$-trivial monoids of Section 5, one can find polynomial times exponential bounds for the rates of convergences after ℓ steps of the form $c\, \ell^k \lambda^{\ell-k}$, where c is the number of chambers, $\lambda = \max_i(1 - x_i)$, and k is a parameter associated to the poset. It is worth pursuing better bounds for the rates of convergence.

In this paper, we have characterized posets, where the Markov chains for the promotion graph yield certain simple formulas for their eigenvalues and multiplicities. The eigenvalues have explicit expressions for rooted forests and there is an explicit combinatorial interpretation for the multiplicities as derangement numbers of permutations for unions

of chains by Theorem 5.4. However, we have not covered all possible posets, whose promotion graphs have nice properties. For example, the non-zero eigenvalues of the transition matrix of the promotion graph of the poset in Example 3.1 are given by

$$-x_1-x_2, \quad -x_1-x_2-x_4, \quad -x_1-x_2-x_3-x_4 \text{ and } -2x_1-x_2-x_3-x_4 \, ,$$

even though the corresponding monoid is not $\mathcal{R}$-trivial (in fact, it is not even aperiodic). On the other hand, not all posets have this property. In particular, the poset with covering relations $1 < 2, 1 < 3$ and $1 < 4$ has six linear extensions, but the characteristic polynomial of its transition matrix does not factorize at all. It would be interesting to classify all posets with the property that all the eigenvalues of the transition matrices of the promotion Markov chain are linear in the probability distribution x_i. In such cases, one would also like an explicit formula for the multiplicity of these eigenvalues. In this paper, this was only achieved for unions of chains.

Appendix A. Sage and Maple implementations

We have implemented the extended promotion and transposition operators on linear extensions in `Maple` and also the open source software `Sage` [S+12, SCc12]. The `Maple` code is available from the homepage of one of the authors (A.A.) as well as the preprint version on the arXiv, whereas the `Sage` code was already integrated into `sage-5.0` (by A.S.). Some of the figures in this paper were produced in `Sage`.

Here is an illustration on how to reproduce the examples of Section 3 in `Sage`. The poset and linear extensions of Example 3.1 can be constructed as follows:

```
sage: P = Poset(([1,2,3,4],[[1,3],[1,4],[2,3]]))
sage: L = P.linear_extensions()
sage: L.list()
[[2, 1, 3, 4], [2, 1, 4, 3], [1, 2, 3, 4], [1, 2, 4, 3],
 [1, 4, 2, 3]]
```

To compute the generalized promotion operator on this poset, using the algorithm defined in Section 2.1, we first need to make sure that the poset P is associated with the identity linear extension:

```
sage: P = P.with_linear_extension([1,2,3,4])
```

Alternatively, this is achieved via

```
sage: P = Poset(([1,2,3,4],[[1,3],[1,4],[2,3]]),
                linear_extension = True)
sage: Q = P.promotion(i=2)
sage: Q.show()
```

The various graphs of Sections 3.1–3.4 can be created and viewed, respectively, as follows:

```
sage: G = L.markov_chain_digraph(action='tau')
sage: G = L.markov_chain_digraph(action='tau',
                labeling='source')
sage: G = L.markov_chain_digraph(action='promotion')
sage: G = L.markov_chain_digraph(action='promotion',
                labeling='source')
sage: view(G)
```

The transition matrices can be computed via

```
sage: L.markov_chain_transition_matrix(action='tau')
```

with again other settings for "action" or "labeling", depending on the desired graph.

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